Geometric grid permutation classes and regular languages

Ruth Hoffmann

University of St Andrews, School of Computer Science

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Definition

A **automaton** is a quintuple $(Q, \Sigma, i, A, \theta)$ where:
Q - set of states;
Σ - alphabet;
$i \in Q$ - start state;
$A \subseteq Q$ - set of accept states;
$\theta : Q \times \Sigma \rightarrow Q$ - transition function

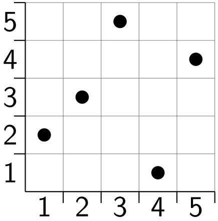
Background

Definition

A **transducer** is a six-tuple $(Q, \Sigma, \Gamma, i, A, \theta)$ where:
Q - set of states;
Σ - input alphabet;
Γ - output alphabet;
$i \in Q$ - start state;
$A \subseteq Q$ - set of accept states;
$\theta : Q \times (\Sigma \cup \{\epsilon\}) \to Q \times (\Gamma \cup \{\epsilon\})$ - transition function, where:
ϵ - empty string.

Background

$$\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 2 & 3 & 5 & 1 & 4\end{pmatrix} = 23514$$



Background

Definition (Classic permutation pattern inclusion)

$\sigma \in S_n$ **contains** or **involves** $\pi \in S_k$, $\pi \leq \sigma$, if σ contains a subsequence of length k order isomorphic to π.

Definition (Classic permutation pattern classes)

A **permutation class** C is a down set of permutations under the containment or involvement order. So if $\sigma \in C$ and $\pi \leq \sigma$ then $\pi \in C$.

Background

Definition (Antichain or Basis)

For any permutation class C, the **antichain** or **basis** B is the set of minimal permutations not in C. So
$C = Av(B) = \{\sigma : \beta \nleq \pi \text{ for all } \beta \in B\}$.

Definition (Partial well-order)

A poset is said to be **partially well-ordered** (pwo) if it contains neither an infinite strict descending chain nor an infinite antichain.

Background

Matrices M are indexed with columns (Left $\rightarrow$ Right), rows (Bottom $\rightarrow$ Top).
M is an $s \times t$ matrix with $0, +1, -1$ as its entries.

Definition (Partial Multiplication Matrix)

M is said to be a **partial multiplication matrix** (pmm) if there are column signs $c_1, \ldots, c_s \in \{\pm 1\}$ and row signs $r_1, \ldots, r_t \in \{\pm 1\}$ such that for all k, ℓ, $M_{k,\ell}$ is equal to either $c_k * r_\ell$ or 0.

Background

Definition (Gridding of a permutation)

An M-**gridding** of a permutation $\sigma \in S_n$ is a pair of sequences $0 = x_1 \leq \ldots \leq x_{s+1} = n$ (column divisors) and $0 = y_1 \leq \ldots \leq y_{t+1} = n$ (row divisors) such that the $\sigma((x_k, x_{k+1}]) \in (y_\ell, y_{\ell+1}]$ are increasing if $M_{k,\ell} = 1$, decreasing if $M_{k,\ell} = -1$ and empty if $M_{k,\ell} = 0$.

Definition (Monotone Grid Class)

The **monotone grid class** or **grid class** of M, $\mathrm{Grid}(M)$, contains all permutations which possess an M-gridding.

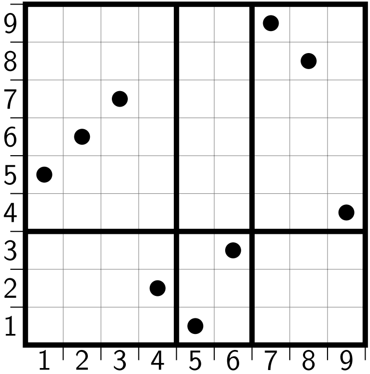
Example

$$M = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

with

$$c_1 = -1, c_2 = c_3 = 1;$$

$$r_1 = 1, r_2 = -1.$$



567213984 possesses an M-gridding.

Geometric Grid Class

Definition (Standard Figure)

The **standard figure** of M is the point set in $\mathbb{R}^2$ consisting of the increasing line segment from $(k-1, \ell-1)$ to (k, ℓ) if $M_{k,\ell} = 1$ or the decreasing line segment from $(k-1, \ell)$ to $(k, \ell-1)$ if $M_{k,\ell} = -1$.

Definition (Geometric Grid Class)

The **geometric grid class** of M, $\mathrm{Geom}(M)$, is the set of all permutations that can be drawn on the standard figure of M as follows:
Choose n points in the figure, no two on a common horizontal or vertical line. Then label the points from 1 to n bottom $\rightarrow$ top and record the labels left $\rightarrow$ right.

Definition (The Grid Language)

Let M be pmm with column signs $c_1, \ldots, c_s$ and row signs, $r_1, \ldots, r_t$ and let $\Sigma = \{(k, \ell) : M_{k,\ell} \neq 0\}$. The map $E^r : \Sigma^* \to \text{Grid}(M)$ describes how to create an M-gridded permutation $\pi = E^r(w)$ or $E(\pi) = w$, for each word $w \in \Sigma^*$, in which the letter (k, ℓ) corresponds to an entry in the (k, ℓ) cell. π is build by reading w left to right inserting the elements as follows:

c_k	r_ℓ	order
1	1	L→R & B→T ↗
1	-1	L→R & T→B ↘
1	1	R→L & B→T ↖
-1	-1	R→L & T→B ↙

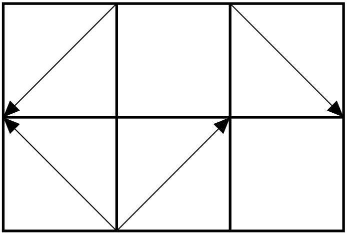
Example

$$M = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$c_1 = -1, c_2 = c_3 = 1;$$

$$r_1 = 1, r_2 = -1.$$

$$\Sigma = \{(1,1),(1,2),(2,1),(3,2)\}$$



$$w = (1,1)(2,1)(1,1)(1,2)(3,2)(3,2)(2,1)$$

Example

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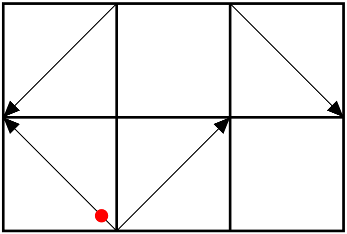

$$w = \color{red}{(1,1)}\color{black}(2,1)(1,1)(1,2)(3,2)(3,2)(2,1)$$

Example

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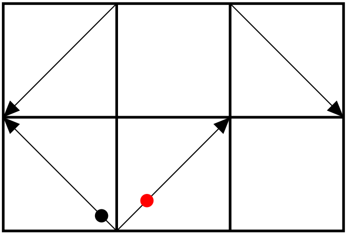
$$\Sigma = \{(1,1), (1,2), (2,1), (3,2)\}$$



$$w = (1,1)\color{red}{(2,1)}(1,1)(1,2)(3,2)(3,2)(2,1)$$

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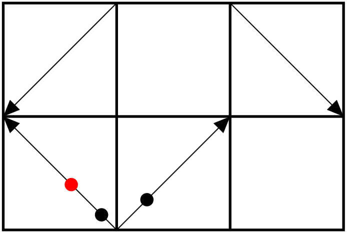

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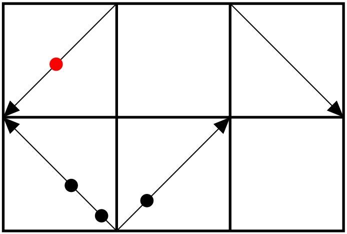

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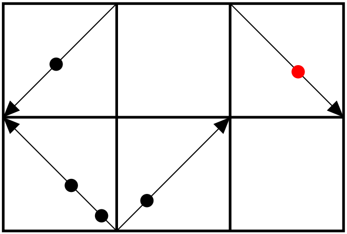

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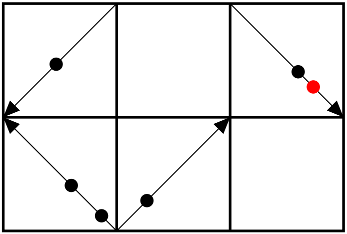

$$w = (1,1)(2,1)(1,1)(1,2)(3,2)\color{red}{(3,2)}(2,1)$$

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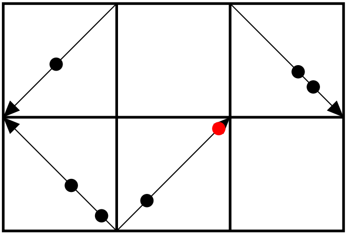
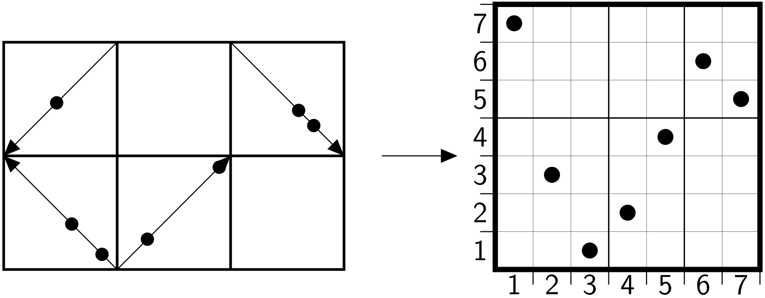


$$w = (1,1)(2,1)(1,1)(1,2)(3,2)(3,2)(2,1)$$

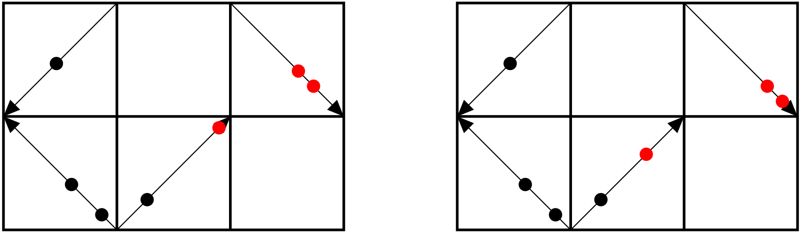
Example



$$E(\pi) = w = (1,1)(2,1)(1,1)(1,2)(3,2)(3,2)(2,1)$$

$$\pi = 7312465$$

Geometric Grid Class



$$w = (1,1)(2,1)(1,1)(1,2)\color{red}{(3,2)(3,2)(2,1)}$$

$$w' = (1,1)(2,1)(1,1)(1,2)\color{red}{(2,1)(3,2)(3,2)}$$

Theorem

Every geometrically griddable class is finitely based.

Original Proof

See (Albert, Atkinson, Bouvel, Ruškuc, Vatter 2011) for details.

Proposition

If the class $\mathcal{C}$ is geometrically griddable, then the class $\mathcal{C}^{+1}$ (one point extension) is also geometrically griddable.

Theorem

Every geometrically griddable class is pwo.

Theorem (Higman's Theorem)

The set of words over any finite alphabet is pwo under the subword order.

Subword-closed languages are regular languages.

$$M_{ij} = 0 \Rightarrow (M_{ij})^{+1} = \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{matrix}$$

$$M_{ij} = 1 \Rightarrow (M_{ij})^{+1} = \begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}$$

$$M_{ij} = -1 \Rightarrow (M_{ij})^{+1} = \begin{matrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \mathbf{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1. \end{matrix}$$

Alternative 1

Proposition

Let $\mathcal{C}$ be a geometric grid class of M. The set of geometric grid encoded basis elements of $\mathcal{C}$ with respect to the matrix M^{+1} is included in

$$E^{M^{+1}}(\mathcal{X}) = \dot{\mathcal{E}}^{M^{+1}}(\mathcal{C}^{+1}) \setminus \dot{E}^{+1}(\mathcal{C}).$$

Alternative 1
Concept of Proof

$E(\mathcal{C})$

(1) ↓ normalise

$\overline{E}(\mathcal{C})$

(2) ↓ extend

All pre-images of $\overline{E}(\mathcal{C})$ with respect to M^{+1}, $\mathcal{E}^{+1}(\mathcal{C})$

standardise (3) ↙ ↘ (4) one point extension

$\dot{\mathcal{E}}^{M^{+1}}(\mathcal{C}) = \dot{E}_G^{+1}(\mathcal{C})$ $\mathcal{E}^{+1}(\mathcal{C}^{+1})$

$$\dot{\mathcal{E}}^{M^{+1}}(\mathcal{C}) = \dot{E}_G^{+1}(\mathcal{C})$$

$$\mathcal{E}^{+1}(\mathcal{C}^{+1})$$

(5) standardise

$$\dot{\mathcal{E}}^{+1}(\mathcal{C}^{+1}) = \dot{E}_G^{M^{+1}}(\mathcal{C}^{+1})$$

(6) difference

$$E_G^{M^{+1}}(\mathcal{X})$$

Alternative 2

Conjecture

Let M, N be two $0, \pm 1$ matrices, with $\mathrm{Geom}(M) = \mathcal{C}$, $\mathrm{Geom}(N) = \mathcal{D}$. Then $\overline{E}^{M \times N}(\mathcal{C} \cap \mathcal{D})$, the language of normal encodings of permutations in $\mathcal{C} \cap \mathcal{D}$, is regular.

Proposition

Let $\mathcal{C}$ be a geometric grid class of M. The set of geometric grid encoded basis elements of $\mathcal{C}$ with respect to the matrix M^{+1} is

$$E^{M^{+1}}(B) = \overline{(\mathcal{E}^C \mathcal{D}^T)}^C \setminus \overline{E}^{M^{+1}}(\mathcal{C}).$$

$$E_G(\mathcal{C})$$

$$(1)\ \Big\downarrow \text{ extend}$$

$$E_G^{M \times M^{+1}}(\mathcal{C} \cap \mathrm{Geom}(M^{+1})) = E_G^{M \times M^{+1}}(\mathcal{C})$$

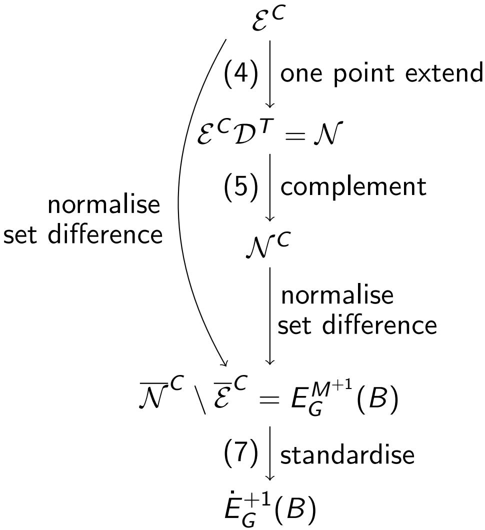
$$(2)\ \Big\downarrow \text{ choose language}$$

$$\mathcal{E} = \{v : (w, v) \in E_G^{M \times M^{+1}}(\mathcal{C})\}$$

$$(3)\ \Big\downarrow \text{ complement}$$

$$\mathcal{E}^C$$

Alternative 2
Concept of Proof



Thanks

Thank you

@ruthhoffmann

`http://ruthh.host.cs.st-andrews.ac.uk`

Literature

- Michael H. Albert, M. D. Atkinson, Mathilde Bouvel, Nik Ruškuc, and Vincent Vatter, *Geometric grid classes of permutations*, ArXiv e-prints (2011), 1–28.
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