Algorithms in grid classes

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Introduction

- Different, computationally efficient representation of permutation pattern classes.
- Solved questions can be constructive, but some are not, yet.
- Want to add more language theoretic aspects to permutation pattern classes.
- This approach could introduce some more constructive approaches to answered or open questions.

Background

Definition

A **transducer** is a six-tuple $(Q, \Sigma, \Gamma, i, A, \theta)$:
Q - set of states;
Σ - input alphabet;
Γ - output alphabet;
$i \in Q$ - start state;
$A \subseteq Q$ - set of accept states;
$\theta : Q \times (\Sigma \cup \{\epsilon\}) \to Q \times (\Gamma \cup \{\epsilon\})$ - transition function, where ϵ - empty word.

Background

Definition (Classic permutation pattern inclusion)

$\sigma \in S_n$ **contains** or **involves** $\pi \in S_k$, $\pi \leq \sigma$, if σ contains a subsequence of length k order isomorphic to π.

Definition (Classic permutation pattern classes)

A **permutation class** C is a down set of permutations under the containment or involvement order. So if $\sigma \in C$ and $\pi \leq \sigma$ then $\pi \in C$.

Definition (Antichain or Basis)

For any permutation class C, the **antichain** or **basis** B is the set of minimal permutations not in C. So
$C = Av(B) = \{\sigma : \beta \not\leq \pi \text{ for all } \beta \in B\}$.

Definition (Partial well-order)

A poset is said to be **partially well-ordered** (pwo) if it contains neither an infinite strict descending chain nor an infinite antichain.

Background

Matrices are indexed with columns (Left $\rightarrow$ Right), rows (Bottom $\rightarrow$ Top).

M is an $s \times t$ matrix with $0, +1, -1$ as its entries.

Definition (Partial Multiplication Matrix [VW11])

M is said to be a **partial multiplication matrix** (pmm) if there are column signs $c_1, \ldots, c_s \in \{\pm 1\}$ and row signs $r_1, \ldots, r_t \in \{\pm 1\}$ such that for all cells (k, ℓ), $M_{k,l}$ is equal to either $c_k * r_l$ or 0.

Background

Definition (Gridding of a permutation [VW11])

An M-**gridding** of a permutation $\sigma \in S_n$ is a pair of sequences $1 = x_1 \leq \ldots \leq x_{s+1} = n + 1$ (column divisors) and $1 = y_1 \leq \ldots \leq y_{t+1} = n + 1$ (row divisors) such that the $\sigma([c_k, c_{k+1})) \in [r_\ell, r_{\ell+1})$ are increasing if $M_{k,\ell} = 1$, decreasing if $M_{k,\ell} = -1$ and empty if $M_{k,\ell} = 0$.

Definition (Monotone Grid Class [VW11])

The **monotone grid class** or **grid class** of M, Grid(M), contains all permutations which possess an M-gridding.

Geometric Grid Class

Definition (Standard Figure [AAB+11])

The **standard figure** of M is the point set in $\mathbb{R}^2$ consisting of the increasing line segment from $(k-1, \ell-1)$ to (k, ℓ) if $M_{k,\ell} = 1$ or the decreasing line segment from $(k-1, \ell)$ to $(k, \ell-1)$ if $M_{k,\ell} = -1$.

Definition (Geometric Grid Class [AAB+11])

The **geometric grid class** of M, Geom(M), is the set of all permutations that can be drawn on the standard figure of M as follows:
Choose n points in the figure, no two on a common horizontal or vertical line. Then label the points from 1 to n bottom $\rightarrow$ top and record the labels left $\rightarrow$ right.

Geometric Grid Class

Definition (The Language [VW11, AAB+11])

Let M be pmm with column signs $c_1, \ldots, c_s$ and row signs, $r_1, \ldots, r_t$ and let $\Sigma = \{(k, \ell) : M_{k,\ell} \neq 0\}$. The map $\varphi : \Sigma^* \to \text{Grid}(M)$ describes how to create an M-gridded permutation $\pi = \varphi(w)$, for each word $w \in \Sigma^*$, in which the letter (k, ℓ) corresponds to an entry in the (k, ℓ) cell. π is build by reading w left to right inserting the elements as follows:

c_k	r_ℓ	order
1	1	L→R & B→T ↗
1	-1	L→R & T→B ↘
1	1	R→L & B→T ↖
-1	-1	R→L & T→B ↙

Theorem

Theorem

Every geometrically griddable class is finitely based.

Original Proof

See [AAB$^+$11] for details.

Proposition

If the class C is geometrically griddable, then the class C^{+1} (one point extension) is also geometrically griddable.

Theorem

Every geometrically griddable class is pwo.

Theorem (Higman's Theorem)

The set of words over any finite alphabet is pwo under the subword order.

Subword-closed languages are regular languages.

Proposition

If the class C is geometrically griddable, then the class C^{+1} (one point extension) is also geometrically griddable.

Take C with M and alphabet Σ. Find C^{+1} with M^{+1} and alphabet Σ^{+1} using transducers.

1. Expand M to $M^{\times 3}$.
2. Find all (finite number) points off the line segments.
3. Use one-point deletion transducer to get the language of the basis.

$$M_{k,\ell} = 0 \Rightarrow M^{\times 3}_{[3*k-2,3*k],[3*\ell-2,3*\ell]} = \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

As $\Sigma = \{(k,\ell) : M_{k,\ell} \neq 0\}$ nothing changes in $\Sigma^{\times 3}$.

$$M_{k,\ell} = 1 \Rightarrow M^{\times 3}_{[3*k-2,3*k],[3*\ell-2,3*\ell]} = \begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

$$(k,\ell) \in \Sigma \Rightarrow$$
$$\{(3*k-2, 3*\ell-2), (3*k-1, 3*\ell-1), (3*k, 3*\ell)\} \in \Sigma^{\times 3}$$

$$M_{k,\ell} = -1 \Rightarrow M^{\times 3}_{[3*k-2,3*k],[3*\ell-2,3*\ell]} = \begin{matrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{matrix}$$

$$(k, \ell) \in \Sigma \Rightarrow$$
$$\{(3*k-2, 3*\ell), (3*k-1, 3*\ell-1), (3*k, 3*\ell-2)\} \in \Sigma^{\times 3}$$

Alternative Proof (Concept)

Transducer 2 $M^{\times 3} \to M^{+1}$

Look at the 3×3 blocks in $M^{\times 3}$:

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \Rightarrow \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x} & 0 & \mathbf{x} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x} & 0 & \mathbf{x} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \qquad \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{array} \Rightarrow \begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ 0 & \mathbf{x} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{x} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}$$

$$\begin{array}{ccc} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{array} \Rightarrow \begin{array}{ccccc} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{x} & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & \mathbf{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}$$

- Based on [AAR03].
- This transducer, $\mathcal{D}$, read words a w (representing permutation π), deletes a letter and returns w' which represents the permutation that is missing the entry corresponding to the deleted letter.
- Applying $\mathcal{D}$ to the language of M^{+1} should result in the language of the basis.

- This transducer, $\mathcal{P}$, deletes multiple letters of the word w (representing π), and returns w' which represents the permutation missing the entries corresponding to the deleted letters.
- In [AAR03] this transducer is then transposed (reversed), so it adds multiple letters and is used to find the language of the class from the language of the basis.

Literature

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