The Regular Language of Simple Permutations

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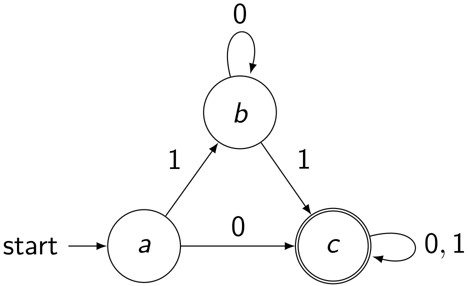
Introduction

- Classical permutation pattern involvement
- Regular Languages
- Simple Permutations
- Block-Decompositions

Regular Languages

- An **alphabet** Σ is a finite set of letters. A **word** is a finite sequence of letters from Σ. Σ^*, the set of all words over Σ. A **language** is a subset of Σ^*.
- **Regular expressions** (RE) are defined as:
  - ε is RE
  - All letters are RE
  - If R_1, R_2 are RE then so are $R_1 R_2$, $R_1 \cup R_2$ and R_1^*.
- A **regular language** is defined by a regular expression.
- A **finite state automaton** (FSA) is a 5-tuple $(\Sigma, S, \delta, q_0, F)$.
- A **regular language** is accepted by a FSA.

Example



$$(0 \cup 10^*1)\Sigma^*$$

Rank Encoding

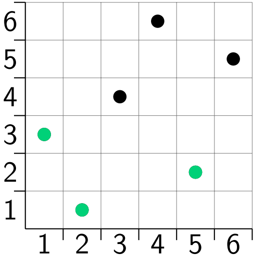
The **rank encoding** of a permutation $\pi = \pi(1) \dots \pi(n)$ is the sequence $E(\pi) = p_1 \dots p_n$ where for all $i \in \{1, \dots, n\}$

$$p_i = |\{x : x \in \{\pi(i), \pi(i+1), \dots, \pi(n)\}, x \leq \pi(i)\}|.$$

[AAR03, ALT97]

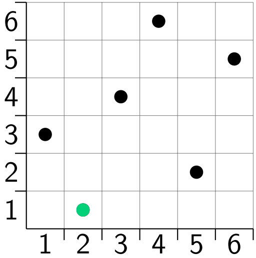
Rank Encoding

$$E(\textcolor{green}{3}14625) = 3$$



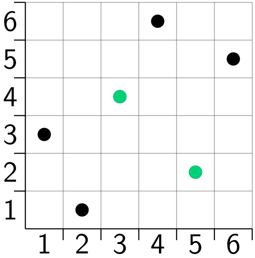
Rank Encoding

$$E(3\textcolor{green}{1}4625) = 31$$

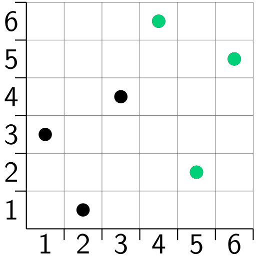


Rank Encoding

$$E(31\textcolor{green}{4}625) = 312$$

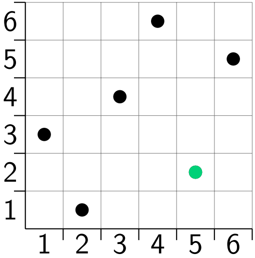


$$E(314\textcolor{green}{6}25) = 3123$$



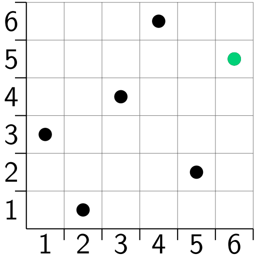
Rank Encoding

$E(314625) = 31231$



Rank Encoding

$$E(31462{\color{green}5}) = 312311$$



Regular Class

In [AAR03] Albert, Atkinson and Ruškuc found that $E(\Omega_k)$ is a regular language, $k \in \mathbb{N}$.
Any subclass of Ω_k is also regular under the rank encoding, and is called a **regular class**.

Simplicity

An **interval** in a permutation is a set of contiguous values, where their set of indices is consecutive.
A permutation π, $|\pi| = n \in \mathbb{N}$ is called **simple** if it only contains intervals of length 1 and n. [Bri10]

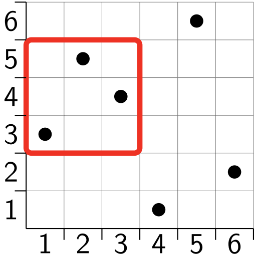


Figure : 354162

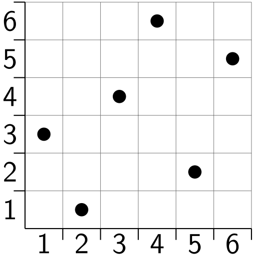
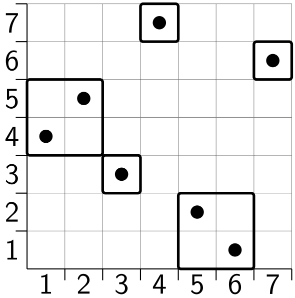


Figure : 314625

A **block-decomposition** of a permutation σ is $\pi[\alpha_1, \ldots, \alpha_m]$, where $|\pi| = m$ and $\alpha_1, \ldots, \alpha_m$ non-empty.

$$4537216 = 32514[12, 1, 1, 21, 1]$$



Block-Decomposition

Let σ be a permutation of finite length greater than 1. There is a unique simple finite permutation π, with $|\pi| > 1$ and a sequence $\alpha_1, \ldots, \alpha_n$ of non-empty permutations such that

$$\sigma = \pi[\alpha_1, \ldots, \alpha_n].$$

If $\pi \neq 12, 21$ then $\alpha_1, \ldots, \alpha_n$ are also uniquely determined by σ. If $\pi = 12$ or 21, then α_1, α_2 are unique so long as we require that α_1 is plus-indecomposable or minus-indecomposable respectively. [AA05]

Languages of block-decomposed permutations

Theorem ([HL13])

Let $\mathcal{C} \subseteq \Omega_k$ be a regular pattern class. The following languages are all regular languages:

1. *The rank encodings of all plus-(in)decomposable permutations in $\mathcal{C}$;*
2. *the rank encodings of all minus-(in)decomposable permutations in $\mathcal{C}$;*

Theorem

Let $\mathcal{C} \subseteq \Omega_k$ be any regular class. Then the set $D_\pi(\mathcal{C}) \subseteq \mathcal{C}$ of π-decomposable permutations of $\mathcal{C}$, for any simple π, with $|\pi| > 2$, is also regular under the rank encoding.

Proof

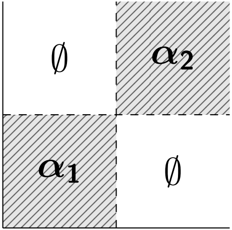


Figure : $12[\alpha_1, \alpha_2]$

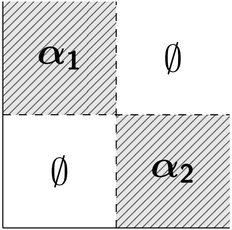
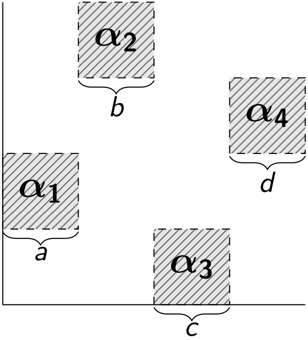


Figure : $21[\alpha_1, \alpha_2]$

q.e.d.

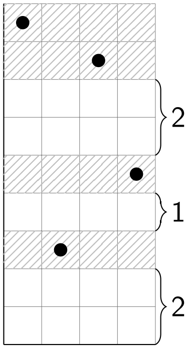
Example

$$2413[\alpha_1, \alpha_2, \alpha_3, \alpha_4]$$



Gaps in Permutations

The **gap sizes** at the end of a prefix of a permutation, is a list of available spaces in the plot of the permutation, between unavailable blocks, read from bottom up.
The part 9385 ends with the gap sizes $\langle 2, 1, 2 \rangle$.



Language of simple permutations

Theorem

The set of all simple permutations of Ω_k is regular under the rank encoding.

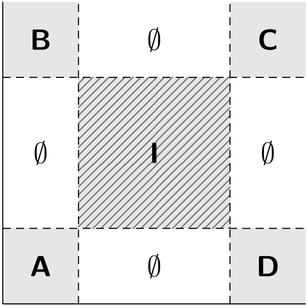
Proof

Language of all non-simple permutations of Ω_k

$$E(\mathcal{NS}_k) = E(\Omega_k) \cap \left(\bigcup_{r=1}^{k-1} P_r \bigcup_{s=r}^{k-1} E(\hat{\Omega}_{k-s})^{+s} \cup \bigcup_{j=1}^{k-1} E(\hat{\Omega}_{k-j})^{+j} \cup \bigcup_{a=2}^{k-1} \bigcup_{b=0}^{k-1-a} Q_{a,b} \bigcup_{i=0}^{a-2} (E(\hat{\Omega}_{k-(b+i)})^{+b+i})^{(a-i)} \right) \Sigma^* \cup E(\mathcal{D}_P(\Omega_k))$$

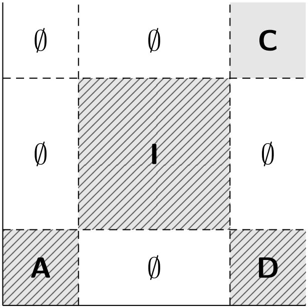
Proof

$$\left(\bigcup_{r=1}^{k-1} P_r \bigcup_{s=r}^{k-1} E(\hat{\Omega}_{k-s})^{+s} \cup \bigcup_{j=1}^{k-1} E(\hat{\Omega}_{k-j})^{+j} \cup \bigcup_{a=2}^{k-1} \bigcup_{b=0}^{k-1-a} Q_{a,b} \bigcup_{i=0}^{a-2} (E(\hat{\Omega}_{k-(b+i)})^{+b+i})^{(a-i)} \right) \Sigma^* \cup E(\mathcal{D}_P(\Omega_k))$$



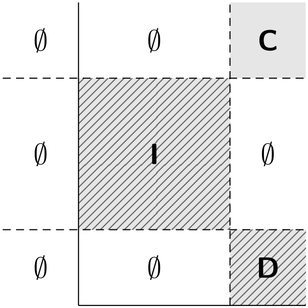
Proof

$$\left(\bigcup_{r=1}^{k-1} P_r \bigcup_{s=r}^{k-1} E(\hat{\Omega}_{k-s})^{+s} \cup \bigcup_{j=1}^{k-1} E(\hat{\Omega}_{k-j})^{+j} \cup \bigcup_{a=2}^{k-1} \bigcup_{b=0}^{k-1-a} Q_{a,b} \bigcup_{i=0}^{a-2} (E(\hat{\Omega}_{k-(b+i)})^{+b+i})^{(a-i)}\right) \Sigma^* \cup E(\mathcal{D}_P(\Omega_k))$$



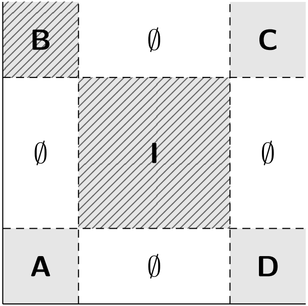
Proof

$$\left(\bigcup_{r=1}^{k-1} P_r \bigcup_{s=r}^{k-1} E(\hat{\Omega}_{k-s})^{+s} \cup \bigcup_{\mathbf{j=1}}^{\mathbf{k-1}} \mathbf{E(\hat{\Omega}_{k-j})^{+j}} \cup \bigcup_{a=2}^{k-1} \bigcup_{b=0}^{k-1-a} Q_{a,b} \bigcup_{i=0}^{a-2} (E(\hat{\Omega}_{k-(b+i)})^{+b+i})^{(a-i)} \right) \boldsymbol{\Sigma^*} \cup E(\mathcal{D}_P(\Omega_k))$$



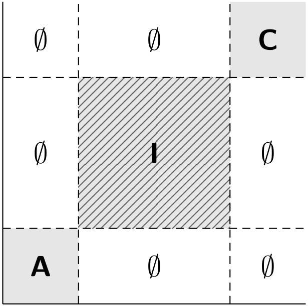
Proof

$$\left(\bigcup_{r=1}^{k-1} P_r \bigcup_{s=r}^{k-1} E(\hat{\Omega}_{k-s})^{+s} \cup \bigcup_{j=1}^{k-1} E(\hat{\Omega}_{k-j})^{+j} \cup \bigcup_{a=2}^{k-1} \bigcup_{b=0}^{k-1-a} \mathbf{Q_{a,b}} \bigcup_{i=0}^{a-2} (E(\hat{\Omega}_{k-(b+i)})^{+b+i})^{(a-i)} \right) \Sigma^* \cup E(\mathcal{D}_P(\Omega_k))$$



Proof

$$\left(\bigcup_{r=1}^{k-1} P_r \bigcup_{s=r}^{k-1} E(\hat{\Omega}_{k-s})^{+s} \cup \bigcup_{j=1}^{k-1} E(\hat{\Omega}_{k-j})^{+j} \cup \bigcup_{a=2}^{k-1} \bigcup_{b=0}^{k-1-a} Q_{a,b} \bigcup_{i=0}^{a-2} (E(\hat{\Omega}_{k-(b+i)})^{+b+i})^{(a-i)} \right) \Sigma^* \cup \mathbf{E}(\boldsymbol{\mathcal{D}}_{\mathbf{P}}(\boldsymbol{\Omega}_{\mathbf{k}}))$$



Proof

$$\left(\bigcup_{r=1}^{k-1} P_r \bigcup_{s=r}^{k-1} E(\hat{\Omega}_{k-s})^{+s} \cup \bigcup_{j=1}^{k-1} E(\hat{\Omega}_{k-j})^{+j} \cup \bigcup_{a=2}^{k-1} \bigcup_{b=0}^{k-1-a} Q_{a,b} \bigcup_{i=0}^{a-2} (E(\hat{\Omega}_{k-(b+i)})^{+b+i})^{(a-i)}\right) \Sigma^* \cup E(\mathcal{D}_P(\Omega_k))$$

Let π be simple and $E(\pi) \in E(\mathcal{NS}_k)$.

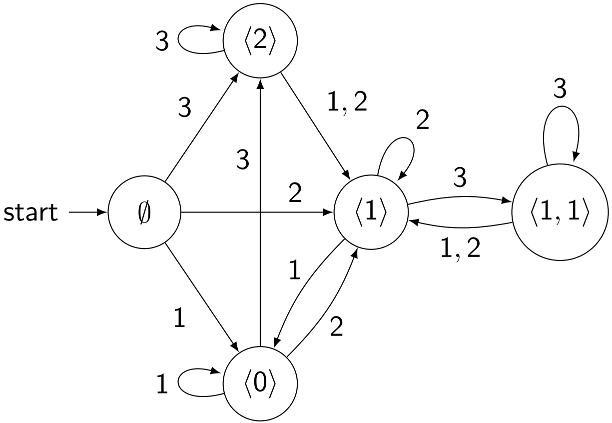
Proof

$$\left(\bigcup_{r=1}^{k-1} P_r \bigcup_{s=r}^{k-1} E(\hat{\Omega}_{k-s})^{+s} \cup \bigcup_{j=1}^{k-1} E(\hat{\Omega}_{k-j})^{+j} \cup \bigcup_{a=2}^{k-1} \bigcup_{b=0}^{k-1-a} Q_{a,b} \bigcup_{i=0}^{a-2} (E(\hat{\Omega}_{k-(b+i)})^{+b+i})^{(a-i)}\right) \Sigma^* \cup E(\mathcal{D}_P(\Omega_k))$$

Are P_r and $Q_{a,b}$ regular languages?

Proof

P_r and $Q_{a,b}$ are based on the gap automaton, only differing in the accept states.



Proof

$E(\mathcal{NS}_k)$ is the regular language of non-simple rank encoded permutations.
So then the set of simple permutations $\mathcal{S}_k \subset \Omega_k$

$$\mathcal{S}_k = \Omega_k \setminus \mathcal{NS}_k$$

$$E(\mathcal{S}_k) = E(\Omega_k \setminus \mathcal{NS}_k) = E(\Omega_k) \setminus E(\mathcal{NS}_k) = E(\Omega_k) \cap E(\mathcal{NS}_k)^C$$

q.e.d.

GAP Implementation [GAP14]

All these automata constructs and more functions are in `PatternClass` for `GAP`. [ALH12] The package can be found on

http://ruthh.host.cs.st-andrews.ac.uk/pkg.html

Thanks

Thank you

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Literature

- M H Albert and M D Atkinson, *Simple permutations and pattern restricted permutations*, Discrete Mathematics **300** (2005), no. 1–3, 1–15.
- M H Albert, M D Atkinson, and N Ruškuc, *Regular closed sets of permutations*, Theoretical Computer Science **306** (2003), no. 1–3, 85–100.
- Michael Albert, Steve Linton, and Ruth Hoffmann, *PatternClass – Permutation Pattern Classes*, http://ruthh.host.cs.st-andrews.ac.uk/pkg.html, 2012.
- M D Atkinson, M J Livesey, and D Tulley, *Permutations generated by token passing in graphs*, Theoretical Computer Science **178** (1997), no. 1–2, 103–118.
- Robert Brignall, *A survey of simple permutations*, Permutation Pattern **376** (2010), 41–65.
- *GAP – Groups, Algorithms, and Programming, Version 4.7.5*, http://www.gap-system.org, 2014.
- Ruth Hoffmann and Steve Linton, *Regular Languages of Plus- and Minus-(In)Decomposable Permutations*, Pure Mathematics and Applications (to appear) (2013).