On Dots in Boxes
or
Permutation Pattern Classes and Regular Languages

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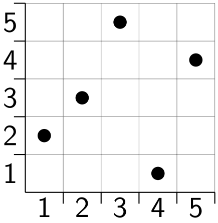
Basics – Regular Language

- $a \in \Sigma$ is a regular language
- ε is a regular language
- $\emptyset$ is a regular language
- if v and w are regular languages then so is $v \cup w$
- if v and w are regular languages then so is vw
- if v is a regular language then so is v^*

Basics – Rank Encoding



$$E(23514) = 22311$$

Summary of Contributions

Proved that the following are regular languages under the rank encoding

- **Plus-(In)Decomposable Permutations**
- Minus-(In)Decomposable Permutations
- Direct Sum of Regular Classes
- σ-Decomposable Permutations
- Simple Permutations

Proved that the following are not regular languages under the rank encoding

- Skew Sum of Regular Classes (in some cases)
- Inflation of Regular Classes (in most cases)
- Wreath Product of Regular Classes (in most cases)
- Wreath Closure of Regular Classes (in nearly all cases)

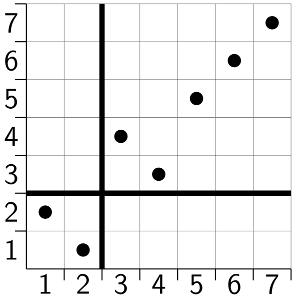
Regularity of Plus-(In)Decomposable Permutations

Theorem (H. and Linton (2013))

Let $\mathcal{C}$ be a regular class. Then $\mathcal{I}_P(\mathcal{C})$, the set of all plus-indecomposable permutations of $\mathcal{C}$, is also regular under the rank encoding.

Corollary (H. and Linton (2013))

Let $\mathcal{C}$ be a regular class. Then $\mathcal{D}_P(\mathcal{C})$, the set of all plus-decomposable permutations of $\mathcal{C}$, is also regular under the rank encoding.



$$E(2143567) = 2121111$$

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Regularity of Minus-(In)Decomposable Permutations

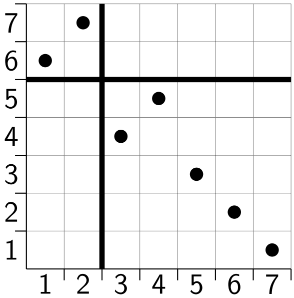
Theorem (H. and Linton (2013))

Let $\mathcal{C}$ be a regular class. Then $\mathcal{D}_M(\mathcal{C})$, the set of all minus-decomposable permutations of $\mathcal{C}$, is also regular under the rank encoding.

Corollary (H. and Linton (2013))

Let $\mathcal{C}$ be a regular class. Then $\mathcal{I}_M(\mathcal{C})$ the set of all minus-indecomposable permutations of $\mathcal{C}$, is also regular under the rank encoding.

Regularity of Minus-(In)Decomposable Permutations



$$E(6745321) = 6644321$$

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Regularity of the Direct and Skew Sum of Classes

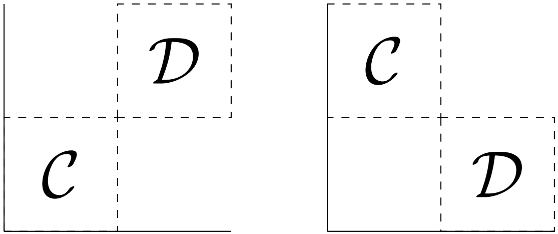
Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be two regular classes under the rank encoding. Then the direct sum $\mathcal{E} = \mathcal{C} \oplus \mathcal{D}$ is a regular class under the rank encoding.

Theorem

Let $\mathcal{C}$ and $\mathcal{D}$ be two regular classes under the rank encoding. Then the skew sum $\mathcal{E} = \mathcal{C} \ominus \mathcal{D}$ is a regular class under the rank encoding if and only if $\mathcal{D}$ is finite.

Regularity of the Direct and Skew Sum of Classes



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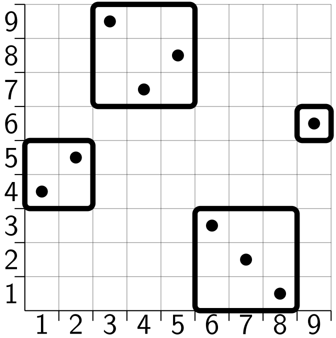
Regularity of σ-decomposable Permutations



Theorem

Let $E(\Omega_k)$ be the regular language of the rank encoded permutations with rank at most k. Let $|\sigma| > 2$ be a simple permutation, then set $\mathcal{D}_\sigma \subseteq \Omega_k$ of σ-decomposable permutations of Ω_k is also regular under the rank encoding.

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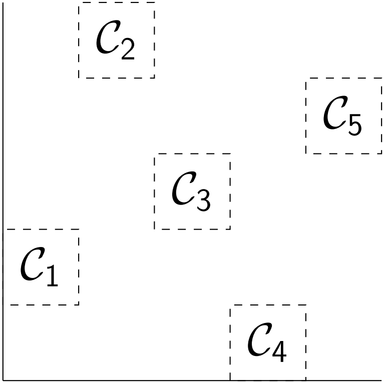
Regularity of the Inflation of Classes

Lemma (34)

Let σ be any simple permutation of length n, $\sigma[\mathcal{C}_1, , \ldots, \mathcal{C}_n] = \mathcal{D}$ be an inflation of σ by the regular classes $\mathcal{C}_i \subseteq \Omega_k$. Then $E(\mathcal{D})$ is a regular language under the rank encoding if and only if $\mathcal{C}_j$ is finite when $\sigma(j)$ is not a left to right maximum.

Regularity of the Inflation of Classes



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Regularity of Simple Permutations

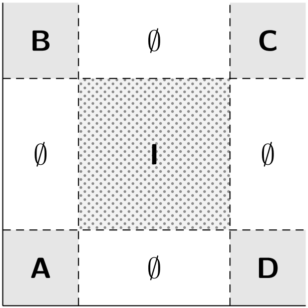
Theorem

The set of all non-simple permutations $\mathcal{NS}_k$ of Ω_k is regular under the rank encoding.

Corollary

The set of simple permutations of Ω_k is regular under the rank encoding.

Regularity of Non-Simple Permutations



Regularity of Non-Simple Permutations

$$
\begin{aligned}
E_R(\mathcal{NS}_k) = & E_R(\Omega_k) \cap \\
& \Bigg(\bigcup_{l=1}^{k-1} \mathscr{P}_l \bigcup_{m=l}^{k-1} m + E_R(\hat{\Omega}_{k-m}) \cup \bigcup_{j=1}^{k-1} j + E_R(\hat{\Omega}_{k-j}) \cup \\
& \bigcup_{a=2}^{k-1} \bigcup_{b=0}^{k-1-a} \mathscr{Q}_{a,b} \bigcup_{i=0}^{a-2} ((b+i) + E_R(\hat{\Omega}_{k-(b+i)}))^{(a-i)} \cup \\
& E_R(\Omega_k \setminus \{\varepsilon\}) E_R(\Omega_k \setminus \{\varepsilon\}) \Bigg) \Sigma^*
\end{aligned}
$$

Simple Permutations of a Non-Regular Class

Theorem

Let $\mathcal{C}$ be a permutation class. Let $Si(\mathcal{C})$ be the set of all simple permutations of $\mathcal{C}$. If $Si(E_k(\mathcal{C})) = Si(E_{k+1}(\mathcal{C})) = Si(E_{k+2}(\mathcal{C}))$ then $Si(E_k(\mathcal{C}))$ is the set of words corresponding to all simple permutations of $\mathcal{C}$.

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Lemma (42)

Let $\mathcal{A}$ and $\mathcal{B}$ be regular languages under the rank encoding then $\mathcal{A} \wr \mathcal{B}$ is regular under the rank encoding if and only if $\mathcal{B}$ is finite or $\mathcal{A} \subseteq Av(21)$.

Regularity of the Wreath Closure of Classes

Lemma (43)

The wreath closure of a regular class $\mathcal{A}$ containing the permutation 21 is not regular.

Corollary

A wreath closed class $\mathcal{A}$ is regular if and only if $\mathcal{A}$ is finite or consists of ascending permutations.

Open Topics

Is there a constructive way of finding the language of the basis of a geometrically griddable class?

Thank you

Thanks!

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