Languages of Permutation Classes and Permutation Subsets

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Motivation

Describing pattern classes in understandable terms is a challenge; usually the avoiding set is used, which can be convoluted and infinite. We are suggesting to use language theoretic tools to characterise some of these classes and their subsets of permutations using the rank encoding. A summary of all these language algorithms has been coded and documented in a `GAP` [1] package called `PatternClass` [2].

Permutations and Pattern Classes

Permutation A **permutation** is a bijective function from $\{1, 2, \ldots, n\}$ to itself, where $n \in \mathbb{N}$.

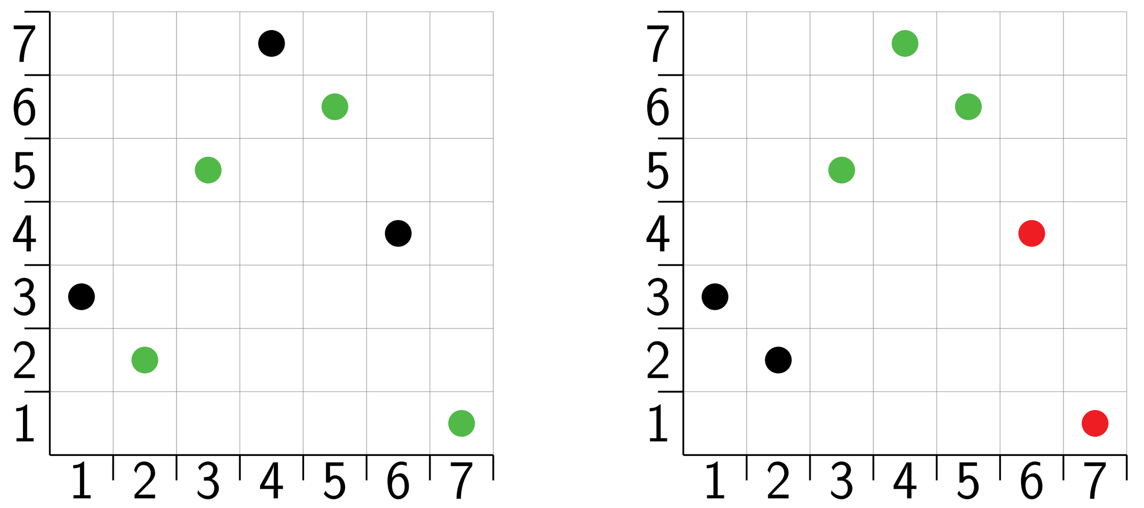
Involvement A permutation π is **involved** in σ, $\pi \preccurlyeq \sigma$, if and only if there is a subsequence in σ that is order isomorphic to π.
$231 \preccurlyeq 31524$ *as* 352 *is order isomorphic to* 231.

Pattern Class A **permutation pattern class** is a set of permutations, closed downwards under the order of involvement. [3]

Rank Encoding The **rank encoding** of a permutation $\pi = \pi_1\pi_2\ldots\pi_n$ is the sequence $E(\pi) = p_1p_2\ldots p_n$ where $p_i = |\{j : j \geq i,\ \pi_j \leq \pi_i\}|$.[4]
The rank encoding of 327986145 *is* 325654111.

Regular Class The class of permutations with maximum rank $k \in \mathbb{N}$, will be denoted as Ω_k. In [4] it was determined that $E(\Omega_k)$ is a regular language. The pattern class $\mathcal{C}$ is said to be a **regular class** if $\mathcal{C} \subseteq \Omega_k$.

Plots of 3257641 identifying 2341 involvement and (in)valid intervals.



Inflation and Block-Decomposition

Interval An **interval** in a permutation is a set of consecutive indices such that the set of their values is contiguous.
In $\pi = 327986145$ $\pi(3)\pi(4)\pi(5) = 798$ *is an interval whereas* $\pi(7)\pi(8)\pi(9) = 145$ *is not.*

Simplicity A permutation of length n is said to be **simple** if it contains intervals of length 0, 1 and n, and no other. [5]
$\pi = 4613572$ *is a simple permutation.* $\sigma = 4635172$ *is not, as* $\sigma(1)\sigma(2)\sigma(3)\sigma(4) = 4635$ *is an interval.*

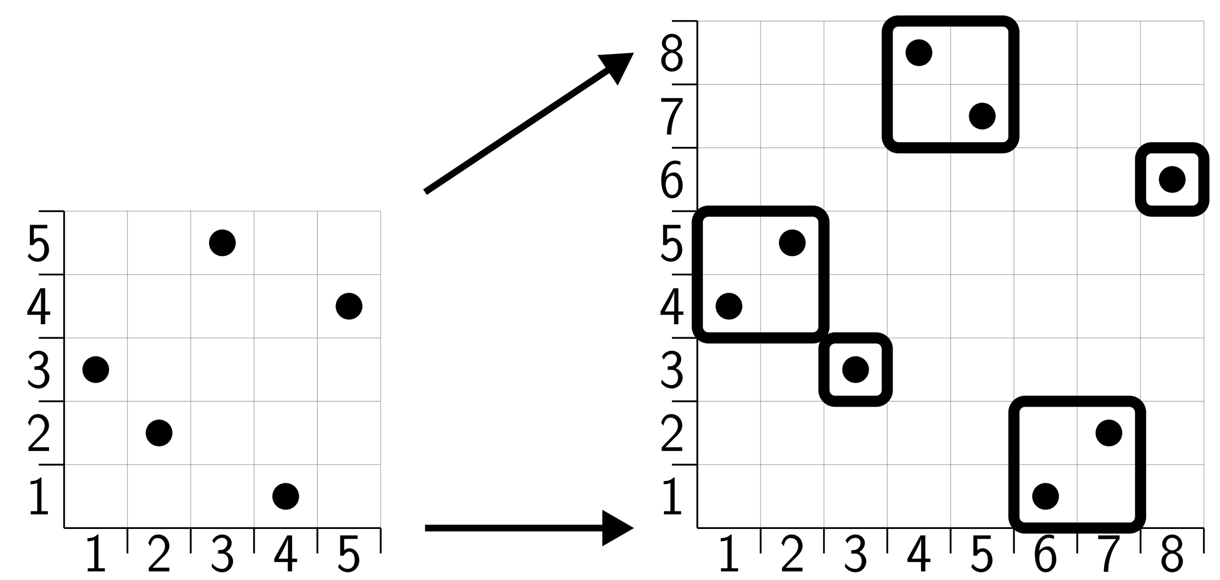
Inflation / Block-Decomposition The **inflation** of a permutation π, $|\pi| = m$, by nonempty permutations $\alpha_1, \ldots, \alpha_m$, $\pi[\alpha_1, \ldots, \alpha_m]$, is the permutation obtained by replacing each entry $\pi(i)$ by an interval that is order isomorphic to α_i.Conversely a **block-decomposition** [3] of σ is any expression of σ as an inflation $\sigma = \pi[\alpha_1, \ldots, \alpha_m]$. [6]
$32514[12, 1, 21, 12, 1] = 45387126$, *is an inflation of* 32514 *and block-decomposition of* 45387126.

Theorem Let $\sigma \in \mathcal{C}$. There is a unique simple permutation $\pi \in \mathcal{C}$ and a sequence $\alpha_1, \ldots, \alpha_n \in \mathcal{C}$ such that

$$\sigma = \pi[\alpha_1, \ldots, \alpha_n].$$

If $\pi \neq 12, 21$ then $\alpha_1, \ldots, \alpha_n$ are also uniquely determined by σ. [3]

Plot of $32514[12, 1, 21, 12, 1] = 45387126$

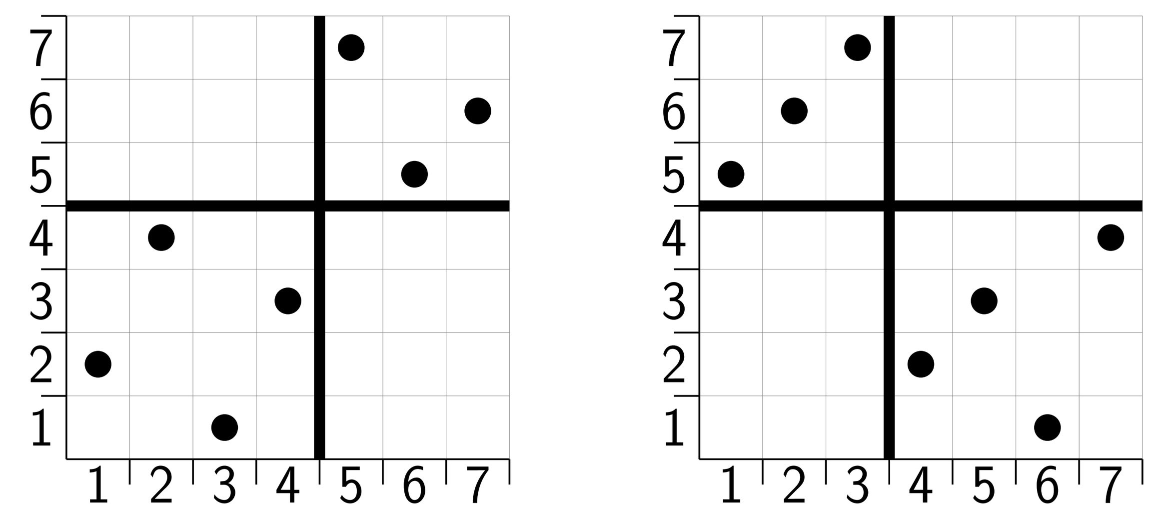


Plus- And Minus-Decomposable Permutations

Plus-Decomposablity A permutation π is $\oplus$*-decomposable* if it can be decomposed as $\pi = 12[\alpha_1, \alpha_2]$.
$2413756 = 12[2413, 312]$ *is* $\oplus$*-decomposable and has rank encoding* $E(2413756) = 2311311$.

Minus-Decomposability A permutation π is $\ominus$*-decomposable* if it can be decomposed as $\pi = 21[\alpha_1, \alpha_2]$.
$5672314 = 21[123, 2314]$ *is* $\ominus$*-decomposable and has rank encoding* $E(5672314) = 5552211$.

Plots of 2413756 and 5672314



⊕/⊖ -Decomposition — Theorem [7]

Let $\mathcal{C}$ be a regular pattern class. The following languages are all regular languages:

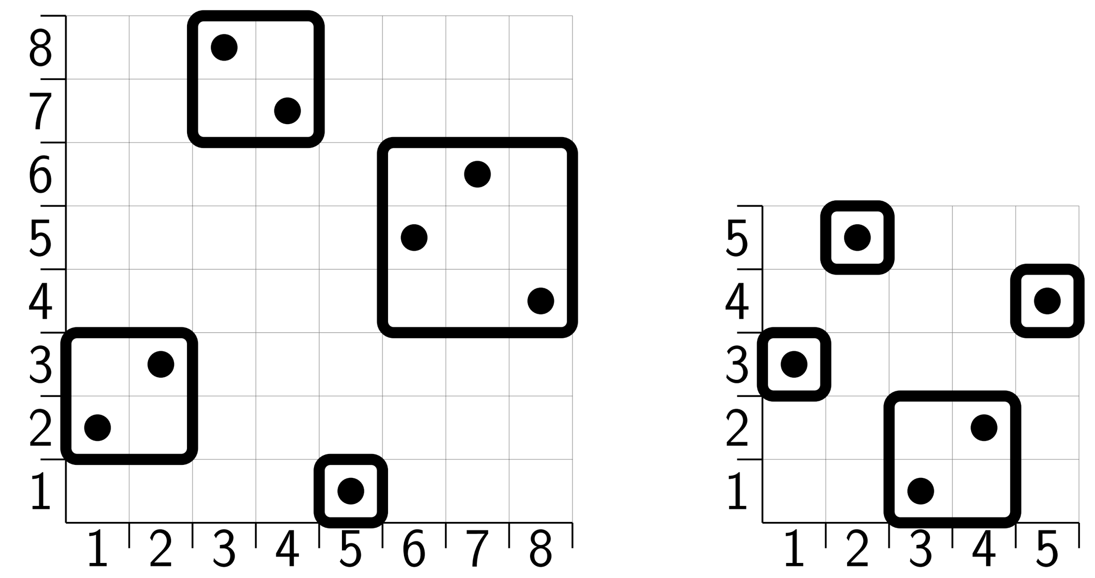
- The rank encodings of all $\oplus$-decomposable permutations in $\mathcal{C}$;
- the rank encodings of all $\oplus$-indecomposable permutations in $\mathcal{C}$;
- the rank encodings of all $\ominus$-decomposable permutations in $\mathcal{C}$;
- the rank encodings of all $\ominus$-indecomposable permutations in $\mathcal{C}$.

Simple Decomposition — Theorem

Let $\mathcal{C} \subseteq \Omega_k$ be any regular class. Then the set $D_\sigma(\mathcal{C}) \subseteq \mathcal{C}$ of σ-decomposable permutations of $\mathcal{C}$, for any simple σ, with $|\sigma| > 2$, is also regular.

For example $2413[12, 21, 1, 231] = 23871564$ *has rank encoding* $E(23871564) = 22651221$, *whereas* $2413[1, 1, 12, 1] = 35124$ *has* $E(35124) = 34111$ *rank encoding.*

Plots of 234981675 and 35124



Simple Permutations — Conjecture

Let $\mathcal{C} \subseteq \Omega_k$ be any regular class. Then the set $S(\mathcal{C}) \subseteq \mathcal{C}$ of simple permutations of $\mathcal{C}$ is also regular.

References

[1] "GAP – Groups, Algorithms, and Programming, Version 4.7.2," 2013. [Online]. Available: http://www.gap-system.org

[2] M. Albert, S. Linton, and R. Hoffmann, "PatternClass – Permutation Pattern Classes," 2012. [Online]. Available: http://www.cs.st-andrews.ac.uk/~ruthh/pkg.html

[3] M. H. Albert and M. D. Atkinson, "Simple permutations and pattern restricted permutations," *Discrete Mathematics*, vol. 300, 2005.

[4] M. H. Albert, M. D. Atkinson, and N. Ruškuc, "Regular closed sets of permutations," *Theoretical Computer Science*, vol. 306, 2003.

[5] R. Brignall, N. Ruškuc, and V. Vatter, "Simple permutations: decidability and unavoidable substructures," *Theoretical Computer Science*, vol. 391, 2008.

[6] R. Brignall, "A survey of simple permutations," *Permutation Pattern*, vol. 376, 2010.

[7] R. Hoffmann and S. Linton, "Regular Languages of Plus- and Minus- (In)Decomposable Permutations," *Pure Mathematics and Applications (to appear)*, 2013.